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ON STABILITY OF SYSTEMS OF GYROSCOPIC STABILIZATION IN THE PRESENCE OF PERTURBATIONS *

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Systems of gyroscopic stabilization regarded here as the electromechanical systems, are considered, with some of the real properties of their elements taken into account. In /1/ the author dealt with the problem of stability of a steady motion of a system of gyroscopic stabilization under parametric perturbations. It is however important that the problem of stability when the pertrubations are continuous is also considered. Using the results of /1/, it can be assumed that small (in a well defined sense) perturbations acting over the electric generalized coordinates and over a part of the mechanical generalized coordinates, should not upset the stability (i.e. they are not significant). This assumption, which requires additional study, is proved in the present paper.

1. We shall consider a system of gyroscopic stabilization using, as in /1/, an electromechanical system on a fixed support as its model. The differential equations of perturbed motion obtained in /1/ have, in the case of the present model, the form

$$\frac{d}{dt}a\mathbf{q}_M^{\cdot} + (b^\circ + g^\circ)\mathbf{q}_M^{\cdot} = \mathbf{Q}_M' + \mathbf{Q}_M'' + \mathbf{\Phi}_M \tag{1.1}$$

$$\frac{d}{dt}L\mathbf{q}_E^{\cdot} + B^\circ\mathbf{q}_3^{\cdot} = \mathbf{Q}_E' + \mathbf{Q}_E'' + \mathbf{\Phi}_E, \quad \frac{d\mathbf{q}_M}{dt} = \mathbf{q}_M$$

$$\mathbf{Q}_M'(\mathbf{q}_M, \mathbf{q}_E^{\cdot}) = \left\|\begin{matrix} 0 \\ A^\circ\mathbf{q}_E^{\cdot} \\ -c^\circ\mathbf{q}_4 \end{matrix}\right\|, \quad \mathbf{Q}_E'(\mathbf{q}_M, \mathbf{q}_E^{\cdot}) = -\left\|\begin{matrix} \omega^\circ\mathbf{q}_1 + R_1^\circ\mathbf{q}_E^{\cdot} \\ (R_2^\circ + \Omega^\circ)\mathbf{q}_E^{\cdot} \\ R_3^\circ\mathbf{q}_E^{\cdot} \end{matrix}\right\|, \quad \mathbf{\Phi}_M = \left\|\begin{matrix} \mathbf{\Phi}_{1M} \\ \mathbf{\Phi}_{2M} \end{matrix}\right\|$$

Here $\mathbf{\Phi}_M = \mathbf{\Phi}_M(\mathbf{q}_M, \mathbf{q}_M^{\cdot}, \mathbf{q}_E^{\cdot}, t)$ and $\mathbf{\Phi}_E = \mathbf{\Phi}_E(\mathbf{q}_M, \mathbf{q}_M^{\cdot}, \mathbf{q}_E^{\cdot}, t)$ are, respectively, n-and u-dimensional vector functions characterizing constantly acting perturbations, and the remaining symbols are those used in /1/. We assume the functions $\mathbf{\Phi}_M$ and $\mathbf{\Phi}_E$ to be such, that a unique solution of (1.1) exists at every point of the region in question.

The zero solution of the system (1.1) without perturbations determines a steady motion, and we shall study its stability. We shall further assume that the perturbations acting on the system lead to the appearance of perturbing forces only with respect to some of the variables. Let us assume that the first m components of the vector $\mathbf{\Phi}_M$ are equal to zero, i.e. $\mathbf{\Phi}_{1M} = 0$. We shall call the system of differential equations (1.1) (under the assumptions made above about the vector $\mathbf{\Phi}_M$) the perturbed system, and a system of equations obtained from (1.1) when $\mathbf{\Phi}_M = 0$ and $\mathbf{\Phi}_E = 0$ the unperturbed system and denote it by (1.1').

We introduce new variables by means of a nonsingular uniformly regular transformation /1/

$$\mathbf{z} = a_1\mathbf{q}_M^{\cdot} + (b_1^\circ + g_1^\circ)\mathbf{q}_M, \quad \mathbf{x}_1 = a\mathbf{q}_M^{\cdot}, \quad \mathbf{x}_2 = L\mathbf{q}_E^{\cdot}, \quad \mathbf{x}_3 = \mathbf{q}_1, \quad \mathbf{x}_4 = \mathbf{q}_4$$

Equations (1.1) in new variables become

$$\frac{d\mathbf{z}}{dt} = \mathbf{Z}, \quad \frac{d\mathbf{x}}{dt} = P\mathbf{x} + \mathbf{X} + \mathbf{\Phi}(t, \mathbf{z}, \mathbf{x}) \tag{1.2}$$

$$P = \left\|\begin{array}{c|c|c|c} -(b^\circ + g^\circ)d^\circ & \begin{matrix} 0 \\ A^\circ L^{-1} \\ 0 \end{matrix} & 0 & \begin{matrix} 0 \\ -c^\circ \end{matrix} \\ \hline -E^\circ & -R^\circ L^{-1} & \begin{matrix} -\omega^\circ \\ 0 \end{matrix} & \\ \hline \begin{matrix} d_1^\circ \\ d_4^\circ \end{matrix} & 0 & 0 & 0 \end{array}\right\|, \quad \mathbf{x} = \left\|\begin{matrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \\ \mathbf{x}_4 \end{matrix}\right\|, \quad R^\circ = \left\|\begin{matrix} R_1^\circ \\ R_2^\circ + \Omega^\circ \\ R_3^\circ \end{matrix}\right\|$$

Here $\mathbf{z}$ and $\mathbf{x}$ are m- and $(2n + u - m)$ -dimensional vectors $\mathbf{Z} = \beta\mathbf{x}_1$, $\mathbf{X} = \gamma\mathbf{x}$ where $\beta = \|\beta_{kj}(\mathbf{z}, \mathbf{x})\|$ and $\gamma = \|\gamma_{rs}(\mathbf{z}, \mathbf{x})\|$ are the $m \times n$ and $(2n + u - m) \times (2n + u - m)$ matrices respectively. The unperturbed system (1.1') will have, in the new variables, a corresponding system (1.2') obtained from (1.2) by setting $\mathbf{\Phi} = 0$.

We shall call the variable $\mathbf{x}$ the basic and variable $\mathbf{z}$ the critical, and

*Prikl.Matem.Mekhan.,44,No.1,172-175,1980 consider the problem of stability of the null solution of the unperturbed system under constantly acting perturbations with respect to some of the variables (with respect to the basic variable). We note that for the system in question a (Liapunov /2/) critical case of m zero roots exists.

2. Let us make the assumption usually made in the course of solving the problems of stability under constantly acting perturbations /3,4/, that the perturbations are small: $\|\Phi\| < \rho$ where $\rho > 0$ is small. Then the following theorem holds.

Theorem 1. If P in the system (1.2) is a stable matrix, then for any given $\varepsilon > 0$ numbers η and ρ greater than zero can be found such, that for any solution of the system (1.2) with initial conditions

$$\|z(t_0)\| < \eta, \quad \|x(t_0)\| < \eta$$

for all $t \geqslant t_0$, the inequalities

$$\|z\| < \varepsilon, \quad \|x\| < \varepsilon$$

will hold for any values of Φ satisfying, in the region in question, the relations

$$\|\Phi\| < \rho \tag{2.1}$$

Proof. Let $\varepsilon > 0$ be given. Consider the solution of the equation

$$d\mathbf{x} / dt = P\mathbf{x} + \mathbf{X}(t, \boldsymbol{\zeta}(t), \mathbf{x}) + \Phi(t, \boldsymbol{\zeta}(t), \mathbf{x}) \tag{2.2}$$

obtained from the equation of the system (1.2) for the basic variable, by making the substitution $z = \zeta(t)$ where $\zeta(t)$ is an arbitrary continuous function with values belonging to the region in question. We also assume that

$$\|\zeta(t_0)\| < \eta, \quad \|x(t_0)\| < \eta, \quad 0 < \eta < \varepsilon$$
$$\|\zeta(t)\| \leqslant \varepsilon, \quad t \in [t_0, t_1]$$

The inequality $\|x(t)\| < \varepsilon$ continues to hold for the values of t near to t_0, by virtue of the continuity. Let $\|x(t_1')\| = \varepsilon$ at some instant of time $t_1' \leqslant t_1$, and let

$$\|\mathbf{X}\| \leqslant \gamma_1(\varepsilon)\varepsilon \tag{2.3}$$

for $t \in [t_0, t_1']$ where $\gamma_1 \to 0$ as $\varepsilon \to 0$. We note that by virtue of the condition of the theorem, $\|e^{Pt}\| \leqslant De^{-\alpha t}$ where D and $\alpha > 0$ are constants. In this case the following relation holds for any solution of (2.2) with initial conditions t_0 and x_0 /5,6/:

$$\|\mathbf{x}\| \leqslant \|\mathbf{x}_0\| De^{-\alpha(t-t_0)} + \int_{t_0}^{t} De^{-\alpha(t-\tau)}\|\mathbf{X}\| d\tau + \left\| \int_{t_0}^{t} e^{P(t-\tau)} \Phi \, d\tau \right\| \tag{2.4}$$

Using the relations (2.1), (2.3) and (2.4), we write the following estimate for $x(t_1')$:

$$\|x(t_1')\| < \eta D + D\gamma_1\varepsilon / \alpha + D\rho / \alpha$$

Selecting $\eta \leqslant \varepsilon / (4D)$, $D\gamma_1 / \alpha \leqslant 1/4$, $D\rho / \alpha \leqslant \varepsilon / 4$, we obtain

$$\|x(t_1')\| < 3/4\, \varepsilon < \varepsilon$$

which is a contradiction. Therefore $\|x(t)\| < \varepsilon$ for all t for which $\|\zeta(t)\| \leqslant \varepsilon$.

Let us now consider the equation

$$dz / dt = \mathbf{Z}$$

Integrating this equation and remembering that the right hand part in the old variables has the form

$$\varkappa(\mathbf{q}_M, \mathbf{q}_M^{\cdot})\mathbf{q}_M^{\cdot}$$

where $\varkappa(\mathbf{q}_M, \mathbf{q}_M^{\cdot})$ is a $m \times n$ matrix, we obtain

$$\|z\| \leqslant \|z_0\| + \left\| \int_{t_0}^{t} \varkappa(\mathbf{q}_M(t), \mathbf{q}_M^{\cdot}(t)) d\mathbf{q}_M(t) \right\| \tag{2.5}$$

Consider the solution of the perturbed system (1.2)

$$\mathbf{x} = \mathbf{x}(t), \quad \mathbf{z} = \mathbf{z}(t)$$

with initial conditions $\|x(t_0)\| < \eta$, $\|z(t_0)\| < \eta$, $0 < \eta < \varepsilon$. By virtue of the continuity, the inequality $\|z(t)\| < \varepsilon$ will still hold for the instances of time t neat t_0. Let the following relation hold at some instant of time t_2:

$$\|z(t_2)\| = \varepsilon \tag{2.6}$$

In accordance with the previous arguments we have, for $t \in [t_0, t_2]$,

$$\|x(t)\| < \varepsilon$$

The following inequalities will hold for the above values of t by virtue of the variable transformation:

$$\|\mathbf{q}_i\| < \varepsilon \quad (i = 1,4); \ \|\mathbf{q}_j\| < \varepsilon_1(\varepsilon) \ (j = 2, 3); \ \|\varkappa\| < \varkappa_1(\varepsilon)$$

where ε_1 and $\varkappa_1 \to 0$ as $\varepsilon \to 0$. Keeping this in mind, we can use (2.5) to obtain the estimate

$$\|z(t_2)\| < \eta + \varkappa_1 M\varepsilon$$

where M is a constant independent of ε.

When $\eta \leqslant \varepsilon/3$, $\varkappa_1 M \leqslant 1/3$ we have $\|z(t_2)\| < 2/3\,\varepsilon < \varepsilon$, which contradicts the assumption (2.6). Thus if $\eta \leqslant \min[\varepsilon / (4D), \varepsilon / 3]$, $\rho \leqslant \alpha\varepsilon / (4D)$, then we have $\|z\| < \varepsilon$, $\|x\| < \varepsilon$ for all $t \geqslant t_0$.

3. Let the perturbations no longer be small, but be bounded functions satisfying the

conditions /7/

$$\|\Phi\| \leqslant K_1, \quad \int_t^{t+T} \Phi(\tau, \bar{x}, \bar{z})\, d\tau = 0, \quad T > 0 \tag{3.1}$$

for any fixed values $x = \bar{x}$ and $z = \bar{z}$.

Theorem 2. If P is a stable matrix in the system (1.2), then for any given value of $\varepsilon > 0$ numbers η and T_0 greater than zero can be found such, that for any solution of the system (1.2) with initial conditions

$$\|z(t_0)\| < \eta, \quad \|x(t_0)\| < \eta$$

the inequalities

$$\|z\| < \varepsilon, \quad \|x\| < \varepsilon$$

will hold for all $t > t_0$ and for any Φ satisfying the conditions (3.1), provided that $T \leqslant T_0$.

Proof. Repeating the previous arguments, we obtain the estimate (2.4) for the solution of (2.2). Remembering that

$$\left\| e^{Pt} \int_{t_0}^{t} e^{-P\tau} \Phi\, d\tau \right\| < KT$$

we obtain from (2.4), in the present case,

$$\|x(t_1')\| < \eta D + D\gamma_1 \varepsilon / \alpha + KT$$

For $\eta \leqslant \varepsilon / (4D)$, $D\gamma_1 / \alpha \leqslant 1/4$, $T \leqslant \varepsilon / (4K)$, we have $\|x(t_1')\| < 3/4\, \varepsilon < \varepsilon$

Repeating the remaining arguments we find that of $\eta \leqslant \min[\varepsilon / 3, \varepsilon / (4D)]$, $T \leqslant T_0 = \varepsilon / (4K)$, then $\|x\| < \varepsilon$, $\|z\| < \varepsilon$ for all $t \geqslant t_0$. The theorem is proved.

4. We note that under the conditions of the theorem the null solution of the unperturbed system (1.2') is stable, but not asymptotically stable /2/. The results obtained imply that perturbations acting constantly with respect to the basic variable (small perturbations or those satisfying the conditions (3.1)) do not affect this stability (in the sense of the statement of the theorem).

Returning to the original variable, we obtain

Theorem 3. If, apart from m zero roots, all remaining roots of the characteristic equation of the first approximation to the unperturbed system (1.1') have negative real parts, then, for any arbitrarily small $\varepsilon > 0$, numbers η and ρ (or η and T_0) exist such that for any solution of the system (1.1) with initial conditions

$$\|q_M(t_0)\| < \eta, \quad \|q_M^{\cdot}(t_0)\| < \eta, \quad \|q_E^{\cdot}(t_0)\| < \eta$$

and for all $t \geqslant t_0$, the inequalities

$$\|q_M\| < \varepsilon, \quad \|q_M^{\cdot}\| < \varepsilon, \quad \|q_E^{\cdot}\| < \varepsilon$$

will hold for any Φ_{2M}, Φ_E satisfying the conditions of the type (2.1) (or 3.1).

It follows therefore that perturbations of the type discussed above, acting constantly over some of the generalized coordinates (over the mechanical coordinates q_3 and q_4 and the electrical coordinates q_E) do not affect the stability of the steady motion of the system of gyroscopic stabilization.

Note. It follows that for the model of the system of gyroscopic stabilization considered above, small or high frequency periodic perturbations acting, e.g. along the stabilization axes and in electric circuits, are insignificant from the point of view of their effect on the stability of the steady motion. The result remains valid in particular cases when the electric circuits of the servo systems are assumed inertialess and when the characteristic kinetic moments of the gyroscope are constant.

The results obtained imply, in particular, that for the mathematical model of the gyrostabilizer used here and in the cases when the amplitudes of the harmonic perturbing moments are small or their frequencies high, all generalized velocities $q_M^{\cdot}$ and $q_E^{\cdot}$ and the mechanical generalized coordinates q_M remain small during the whole period of motion of the system.

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